# Problem 3.33

**Sequential measurements.** An operator $\hat{A}$, representing observable $A$, has two (normalized) eigenstates $\psi_1$ and $\psi_2$, with eigenvalues $a_1$ and $a_2$, respectively. Operator $\hat{B}$, representing observable $B$, has two (normalized) eigenstates $\phi_1$ and $\phi_2$, with eigenvalues $b_1$ and $b_2$. The eigenstates are related by

$$\psi_1 = (3\phi_1 + 4\phi_2)/5, \quad \psi_2 = (4\phi_1 - 3\phi_2)/5.$$

**(a)** Observable $A$ is measured, and the value $a_1$ is obtained. What is the state of the system (immediately) after this measurement?

**(b)** If $B$ is now measured, what are the possible results, and what are their probabilities?

**(c)** Right after the measurement of $B$, $A$ is measured again. What is the probability of getting $a_1$? (Note that the answer would be quite different if I had told you the outcome of the $B$ measurement.)

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**Solution**

Write down the eigenvalue problems for $\hat{A}$ and $\hat{B}$.

$$\hat{A}|\psi_n\rangle = a_n|\psi_n\rangle, \quad n = 1, 2 \qquad\qquad \hat{B}|\phi_n\rangle = b_n|\phi_n\rangle, \quad n = 1, 2$$

Let the state vector $|\mathcal{S}\rangle$ represent the system prior to measurement. Since observable $A$ is to be measured first, expand $|\mathcal{S}\rangle$ in terms of the eigenstates, $|\psi_1\rangle$ and $|\psi_2\rangle$, using the projection operator.

$$\begin{aligned}
|\mathcal{S}\rangle &= \hat{I}|\mathcal{S}\rangle \\
&= \left(\sum_{n=1}^{2} |\psi_n\rangle\langle\psi_n|\right)|\mathcal{S}\rangle \\
&= \sum_{n=1}^{2} |\psi_n\rangle\langle\psi_n\,|\,\mathcal{S}\rangle \\
&= \sum_{n=1}^{2} |\psi_n\rangle a_n \\
&= \sum_{n=1}^{2} a_n|\psi_n\rangle \\
&= a_1|\psi_1\rangle + a_2|\psi_2\rangle
\end{aligned}$$

Mathematically, $\langle\psi_n\,|\,\mathcal{S}\rangle$ is interpreted as the component of $|\mathcal{S}\rangle$ along $|\psi_n\rangle$. Physically, $\langle\psi_n\,|\,\mathcal{S}\rangle$ is interpreted as the value of observable $A$ measured in the lab as the system changes from state $|\mathcal{S}\rangle$ to state $|\psi_n\rangle$. Whether $a_1$ or $a_2$ is obtained can't be said for certain before the measurement, but the probabilities, $P(a_1)$ and $P(a_2)$, can be computed by using projection operators again.

Begin with the basic formula relating these probabilities with the expectation value of $A$.

$$\begin{aligned}
\sum_{n=1}^{2} a_n P(a_n) = \langle A \rangle &= \frac{\langle \mathcal{S} \,|\, \hat{A} \,|\, \mathcal{S} \rangle}{\langle \mathcal{S} \,|\, \mathcal{S} \rangle} \\
&= \frac{\langle \mathcal{S} \,|\, \hat{I}\hat{A}\hat{I} \,|\, \mathcal{S} \rangle}{(a_1^* \langle \psi_1| + a_2^* \langle \psi_2|)(a_1|\psi_1\rangle + a_2|\psi_2\rangle)} \\
&= \frac{\left\langle \mathcal{S} \middle| \left( \sum_{m=1}^{2} |\psi_m\rangle\langle\psi_m| \right) \hat{A} \left( \sum_{n=1}^{2} |\psi_n\rangle\langle\psi_n| \right) \middle| \mathcal{S} \right\rangle}{a_1^* a_1 \langle \psi_1 \,|\, \psi_1 \rangle + a_1^* a_2 \langle \psi_1 \,|\, \psi_2 \rangle + a_2^* a_1 \langle \psi_2 \,|\, \psi_1 \rangle + a_2^* a_2 \langle \psi_2 \,|\, \psi_2 \rangle} \\
&= \frac{\sum_{m=1}^{2} \sum_{n=1}^{2} \langle \mathcal{S} \,|\, \psi_m \rangle \langle \psi_m | \cdot \left( \hat{A} |\psi_n\rangle \right) \langle \psi_n \,|\, \mathcal{S} \rangle}{a_1^* a_1 + a_2^* a_2} \\
&= \frac{\sum_{m=1}^{2} \sum_{n=1}^{2} \langle \mathcal{S} \,|\, \psi_m \rangle \langle \psi_m | \cdot (a_n |\psi_n\rangle) \langle \psi_n \,|\, \mathcal{S} \rangle}{|a_1|^2 + |a_2|^2} \\
&= \frac{\sum_{m=1}^{2} \sum_{n=1}^{2} a_n \langle \mathcal{S} \,|\, \psi_m \rangle \langle \psi_m \,|\, \psi_n \rangle \langle \psi_n \,|\, \mathcal{S} \rangle}{|a_1|^2 + |a_2|^2} \\
&= \frac{\sum_{m=1}^{2} \sum_{n=1}^{2} a_n \langle \mathcal{S} \,|\, \psi_m \rangle \delta_{mn} \langle \psi_n \,|\, \mathcal{S} \rangle}{|a_1|^2 + |a_2|^2} \\
&= \frac{\sum_{n=1}^{2} a_n \langle \mathcal{S} \,|\, \psi_n \rangle \langle \psi_n \,|\, \mathcal{S} \rangle}{|a_1|^2 + |a_2|^2} \\
&= \frac{\sum_{n=1}^{2} a_n \langle \psi_n \,|\, \mathcal{S} \rangle^* \langle \psi_n \,|\, \mathcal{S} \rangle}{|a_1|^2 + |a_2|^2} \\
&= \frac{\sum_{n=1}^{2} a_n \left| \langle \psi_n \,|\, \mathcal{S} \rangle \right|^2}{|a_1|^2 + |a_2|^2} \\
&= \frac{\sum_{n=1}^{2} a_n |a_n|^2}{|a_1|^2 + |a_2|^2} = \sum_{n=1}^{2} a_n \left( \frac{|a_n|^2}{|a_1|^2 + |a_2|^2} \right)
\end{aligned}$$

Consequently, the probabilities of getting $a_1$ and $a_2$ are

$$P(a_1) = \frac{|a_1|^2}{|a_1|^2 + |a_2|^2} \quad \text{and} \quad P(a_2) = \frac{|a_2|^2}{|a_1|^2 + |a_2|^2}.$$

As a result of measuring the observable $A$ and getting $a_1$, the state vector changes to $|\psi_1\rangle$.

$$|\psi_1\rangle = \frac{3}{5}|\phi_1\rangle + \frac{4}{5}|\phi_2\rangle$$

If $B$ is then measured, there are two possible results, $b_1$ and $b_2$, which lead to the system's state vector changing to $|\phi_1\rangle$ and $|\phi_2\rangle$, respectively. To determine the probabilities of getting $b_1$ and $b_2$, write $\langle B\rangle$ in terms of $|\psi_1\rangle$ and then use the projection operators written in terms of $|\phi_n\rangle$.

$$\begin{aligned}
\sum_{n=1}^{2} b_n P(b_n) = \langle B\rangle &= \frac{\langle \psi_1 \,|\, \hat{B} \,|\, \psi_1\rangle}{\langle \psi_1 \,|\, \psi_1\rangle} \\
&= \frac{\langle \psi_1 \,|\, \hat{I}\hat{B}\hat{I} \,|\, \psi_1\rangle}{\left(\frac{3}{5}\langle\phi_1| + \frac{4}{5}\langle\phi_2|\right)\left(\frac{3}{5}|\phi_1\rangle + \frac{4}{5}|\phi_2\rangle\right)} \\
&= \frac{\left\langle \psi_1 \middle| \left(\sum_{m=1}^{2} |\phi_m\rangle\langle\phi_m|\right) \hat{B} \left(\sum_{n=1}^{2} |\phi_n\rangle\langle\phi_n|\right) \middle| \psi_1 \right\rangle}{\frac{9}{25}\langle\phi_1 \,|\, \phi_1\rangle + \frac{12}{25}\langle\phi_1 \,|\, \phi_2\rangle + \frac{12}{25}\langle\phi_2 \,|\, \phi_1\rangle + \frac{16}{25}\langle\phi_2 \,|\, \phi_2\rangle} \\
&= \frac{\sum_{m=1}^{2}\sum_{n=1}^{2} \langle\psi_1 \,|\, \phi_m\rangle\langle\phi_m| \cdot \left(\hat{B}|\phi_n\rangle\right)\langle\phi_n \,|\, \psi_1\rangle}{\frac{9}{25} + \frac{16}{25}} \\
&= \frac{\sum_{m=1}^{2}\sum_{n=1}^{2} \langle\psi_1 \,|\, \phi_m\rangle\langle\phi_m| \cdot (b_n|\phi_n\rangle)\langle\phi_n \,|\, \psi_1\rangle}{1} \\
&= \sum_{m=1}^{2}\sum_{n=1}^{2} b_n \langle\psi_1 \,|\, \phi_m\rangle\langle\phi_m \,|\, \phi_n\rangle\langle\phi_n \,|\, \psi_1\rangle \\
&= \sum_{m=1}^{2}\sum_{n=1}^{2} b_n \langle\psi_1 \,|\, \phi_m\rangle\delta_{mn}\langle\phi_n \,|\, \psi_1\rangle \\
&= \sum_{n=1}^{2} b_n \langle\psi_1 \,|\, \phi_n\rangle\langle\phi_n \,|\, \psi_1\rangle \\
&= \sum_{n=1}^{2} b_n \langle\phi_n \,|\, \psi_1\rangle^*\langle\phi_n \,|\, \psi_1\rangle \\
&= \sum_{n=1}^{2} b_n |\langle\phi_n \,|\, \psi_1\rangle|^2
\end{aligned}$$

Consequently, the probabilities of getting $b_1$ and $b_2$ are

$$P(b_1) = |\langle \phi_1 \,|\, \psi_1 \rangle|^2 = \left| \frac{3}{5} \langle \phi_1 \,|\, \phi_1 \rangle + \frac{4}{5} \langle \phi_1 \,|\, \phi_2 \rangle \right|^2 = \left| \frac{3}{5} \right|^2 = \frac{9}{25}$$

$$P(b_2) = |\langle \phi_2 \,|\, \psi_1 \rangle|^2 = \left| \frac{3}{5} \langle \phi_2 \,|\, \phi_1 \rangle + \frac{4}{5} \langle \phi_2 \,|\, \phi_2 \rangle \right|^2 = \left| \frac{4}{5} \right|^2 = \frac{16}{25}.$$

If the value $b_1$ is measured, then the state vector changes to $|\phi_1\rangle$. If the value $b_2$ is measured instead, then the state vector changes to $|\phi_2\rangle$. After the measurement of observable $B$, it's desired to measure observable $A$ again. The aim is to compute the probability of getting $a_1$ from the state $|\phi_1\rangle$ and the probability of getting $a_1$ from the state $|\phi_2\rangle$. Due to normalization, $\langle \phi_1 \,|\, \phi_1 \rangle = \langle \phi_2 \,|\, \phi_2 \rangle = 1$.

$$\begin{aligned}
\sum_{n=1}^{2} a_n P(a_n | b_1) = \langle A \rangle &= \frac{\langle \phi_1 \,|\, \hat{A} \,|\, \phi_1 \rangle}{\langle \phi_1 \,|\, \phi_1 \rangle} \\
&= \frac{\langle \phi_1 \,|\, \hat{I}\hat{A}\hat{I} \,|\, \phi_1 \rangle}{1} \\
&= \left\langle \phi_1 \left| \left( \sum_{m=1}^{2} |\psi_m\rangle\langle\psi_m| \right) \hat{A} \left( \sum_{n=1}^{2} |\psi_n\rangle\langle\psi_n| \right) \right| \phi_1 \right\rangle \\
&= \sum_{m=1}^{2} \sum_{n=1}^{2} \langle \phi_1 \,|\, \psi_m \rangle \langle \psi_m | \cdot \left( \hat{A} |\psi_n\rangle \right) \langle \psi_n \,|\, \phi_1 \rangle \\
&= \sum_{m=1}^{2} \sum_{n=1}^{2} \langle \phi_1 \,|\, \psi_m \rangle \langle \psi_m | \cdot (a_n |\psi_n\rangle) \langle \psi_n \,|\, \phi_1 \rangle \\
&= \sum_{m=1}^{2} \sum_{n=1}^{2} a_n \langle \phi_1 \,|\, \psi_m \rangle \langle \psi_m \,|\, \psi_n \rangle \langle \psi_n \,|\, \phi_1 \rangle \\
&= \sum_{m=1}^{2} \sum_{n=1}^{2} a_n \langle \phi_1 \,|\, \psi_m \rangle \delta_{mn} \langle \psi_n \,|\, \phi_1 \rangle
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{2} a_n P(a_n | b_2) = \langle A \rangle &= \frac{\langle \phi_2 \,|\, \hat{A} \,|\, \phi_2 \rangle}{\langle \phi_2 \,|\, \phi_2 \rangle} \\
&= \frac{\langle \phi_2 \,|\, \hat{I}\hat{A}\hat{I} \,|\, \phi_2 \rangle}{1} \\
&= \left\langle \phi_2 \left| \left( \sum_{m=1}^{2} |\psi_m\rangle\langle\psi_m| \right) \hat{A} \left( \sum_{n=1}^{2} |\psi_n\rangle\langle\psi_n| \right) \right| \phi_2 \right\rangle \\
&= \sum_{m=1}^{2} \sum_{n=1}^{2} \langle \phi_2 \,|\, \psi_m \rangle \langle \psi_m | \cdot \left( \hat{A} |\psi_n\rangle \right) \langle \psi_n \,|\, \phi_2 \rangle \\
&= \sum_{m=1}^{2} \sum_{n=1}^{2} \langle \phi_2 \,|\, \psi_m \rangle \langle \psi_m | \cdot (a_n |\psi_n\rangle) \langle \psi_n \,|\, \phi_2 \rangle \\
&= \sum_{m=1}^{2} \sum_{n=1}^{2} a_n \langle \phi_2 \,|\, \psi_m \rangle \langle \psi_m \,|\, \psi_n \rangle \langle \psi_n \,|\, \phi_2 \rangle \\
&= \sum_{m=1}^{2} \sum_{n=1}^{2} a_n \langle \phi_2 \,|\, \psi_m \rangle \delta_{mn} \langle \psi_n \,|\, \phi_2 \rangle
\end{aligned}$$

Continue the simplification.

$$\sum_{n=1}^{2} a_n P(a_n|b_1) = \sum_{n=1}^{2} a_n \langle \phi_1 \,|\, \psi_n \rangle \langle \psi_n \,|\, \phi_1 \rangle \qquad \sum_{n=1}^{2} a_n P(a_n|b_2) = \sum_{n=1}^{2} a_n \langle \phi_2 \,|\, \psi_n \rangle \langle \psi_n \,|\, \phi_2 \rangle$$

$$= \sum_{n=1}^{2} a_n \langle \phi_1 \,|\, \psi_n \rangle \langle \phi_1 \,|\, \psi_n \rangle^* \qquad = \sum_{n=1}^{2} a_n \langle \phi_2 \,|\, \psi_n \rangle \langle \phi_2 \,|\, \psi_n \rangle^*$$

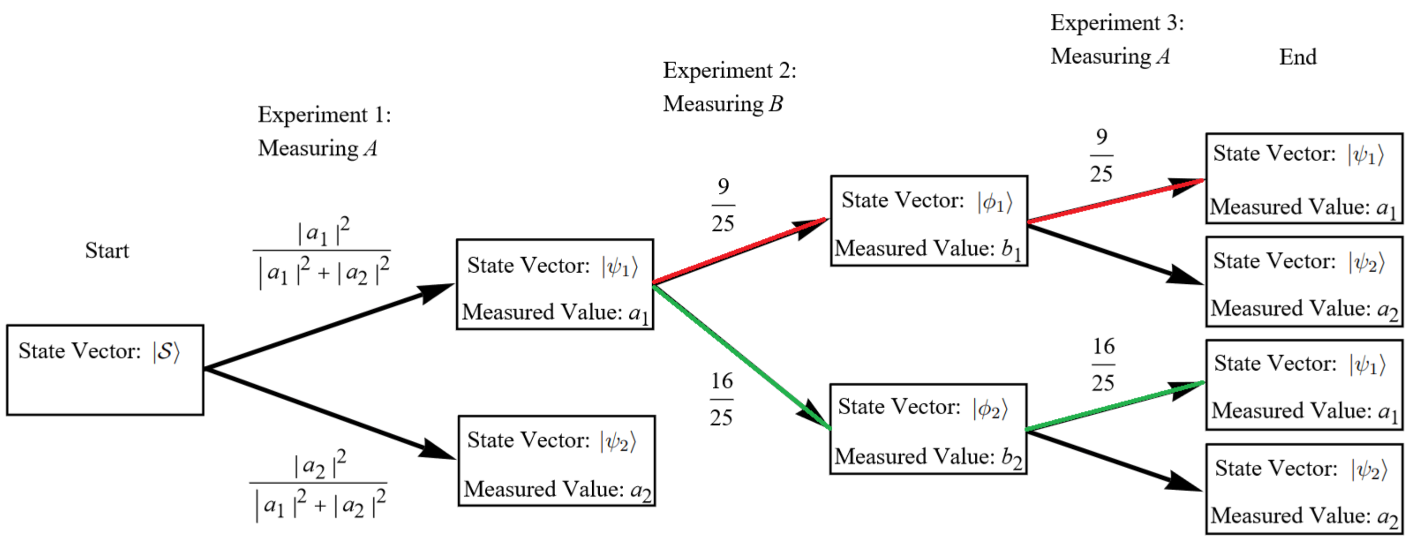
$$= \sum_{n=1}^{2} a_n |\langle \phi_1 \,|\, \psi_n \rangle|^2 \qquad = \sum_{n=1}^{2} a_n |\langle \phi_2 \,|\, \psi_n \rangle|^2$$

Consequently, the probability of measuring $a_1$ from state $|\phi_1\rangle$ and the probability of measuring $a_1$ from state $|\phi_2\rangle$ are respectively

$$P(a_1|b_1) = |\langle \phi_1 \,|\, \psi_1 \rangle|^2 = \left| \frac{3}{5} \langle \phi_1 \,|\, \phi_1 \rangle + \frac{4}{5} \langle \phi_1 \,|\, \phi_2 \rangle \right|^2 = \left| \frac{3}{5} \right|^2 = \frac{9}{25}$$

$$P(a_1|b_2) = |\langle \phi_2 \,|\, \psi_1 \rangle|^2 = \left| \frac{3}{5} \langle \phi_2 \,|\, \phi_1 \rangle + \frac{4}{5} \langle \phi_2 \,|\, \phi_2 \rangle \right|^2 = \left| \frac{4}{5} \right|^2 = \frac{16}{25}.$$

The tree diagram below illustrates the possible outcomes from start to end.




The two highlighted paths show the different (mutually exclusive) ways to get $a_1$ at the end. Use the chain rule for conditional probability to obtain the probability of getting $a_1$ after the measurement of observables, $B$ and $A$.

$$P(b_1)P(a_1|b_1) + P(b_2)P(a_1|b_2) = \frac{9}{25}\frac{9}{25} + \frac{16}{25}\frac{16}{25} = \frac{337}{625} \approx 0.54$$